Logarithms

What is a logarithm?

We can use indices to write $1 = 10^0, 10 = 10^1, 100 = 10^2, 1000 = 10^3\ etc.$

The number 10, used above, is called the base and the power the logarithm to base 10.

We could rewrite the above as

$$log_{10}(1) = 0, \quad log_{10}(10) = 1, \quad log_{10}(100) = 2, \quad log_{10}(1000) = 3$$

Generally, if a is the base we want to use and we can write $y = a^x$, then $x = log_a y$.

It is clear that the rules for logarithms must closely follow those for indices.

If x, y, a and b are positive numbers we can write

$$log_a(xy) = log_a x + log_a y$$

$$log_a\left(\frac{x}{y}\right) = log_a x - log_a y$$

$$log_a x^n = n log_a x$$

Change of base $\qquad log_a x = \frac{log_b x}{log_b a}$

$$log_a 1 = 0$$

$$log_a a = 1$$

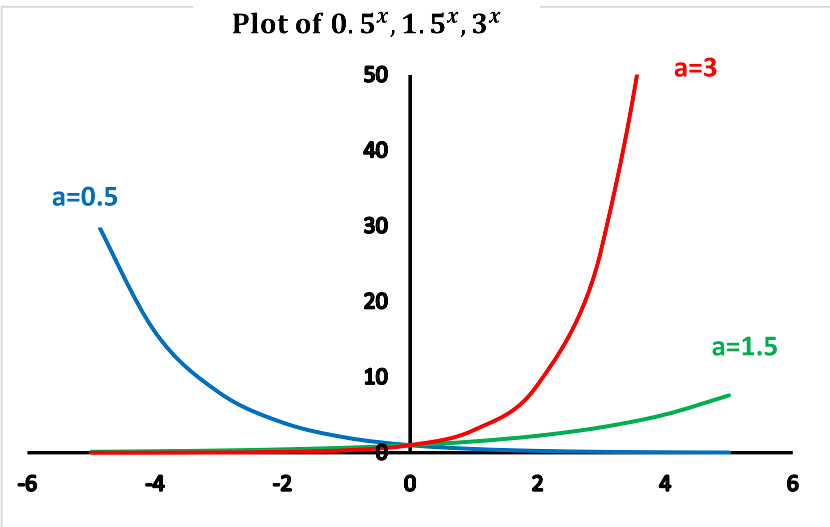
If $\qquad log_a x = log_a y \quad \text{then} \quad x = y$

From the definition if $y = a^x$, then $x = log_a y \quad \text{and therefore} \quad y = a^{log_a y}$

There is one **special** logarithm base. It is **base e**, where **e = 2.7182818** to 7 decimal places. Because of its special status we **do not** use the standard notation of log_e but we use instead $\boldsymbol{ln}$ and so $\boldsymbol{log_e} \equiv \boldsymbol{ln}$.

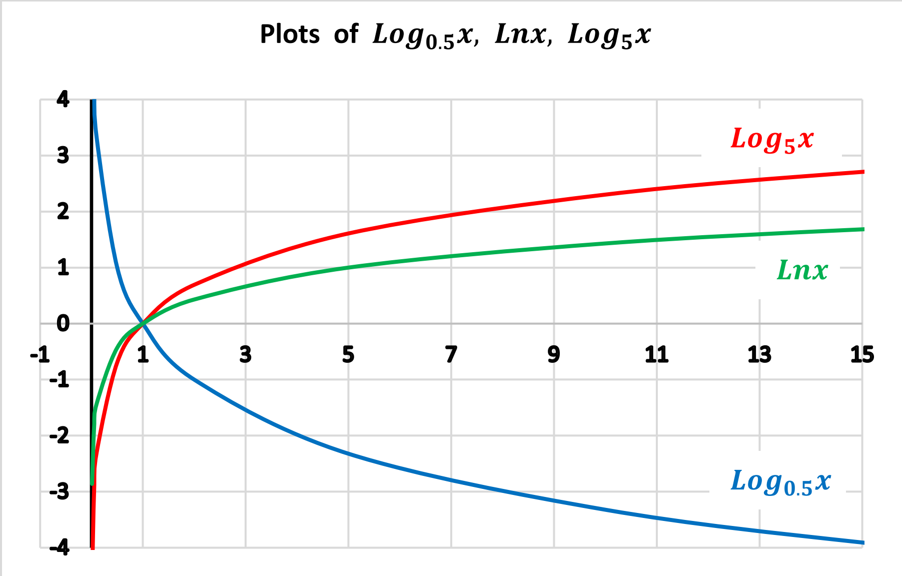
Graph of a^x

Note, the shape of the curve changes for a<1 and a>1. For a>1 it is an increasing function from 0 to ∞ , whereas for a<1 it is a decreasing function from −∞ to 0. For values of a very close to 1 it is almost a horizontal line. It should be noted that all graphs pass through 1 when x = 0.



Graph of $log_a x$

Note, the shape of the curve changes for a<1 and a>1. For a>1 it is an increasing function from $-\infty$ to $+\infty$, whereas for a<1 it is a decreasing function from $+\infty$ to $-\infty$. It should be noted that all graphs pass through 0 when x = 1



Note that lnx is the **inverse function** of e^x and so the plot of e^x can be obtained by reflecting the green curve above, in the line $y = x$.

Solving equations using logarithms

Suppose we have to solve the equation $a^x = b$. If we take the logarithm both sides we can write $\quad log_c(a^x) = log_c(b) \;\; i.e., \;\; xlog_c(a) = log_c(b)$ **or** $x = \frac{log_c(b)}{log_c(a)}$

If $c = a$ then $x = \frac{log_a(b)}{log_a(a)} = log_a(b)$, whereas if $c = e$ then $x = \frac{ln(b)}{ln(a)}$.

Linearising using Logarithms

Suppose we wish to find $a(> 0)$ and n in the equation $y = ax^n$. Taking natural logarithms gives $\quad lny = lna + nlnx$

Thus, if you plot lny against lnx the slope of the line would be n and the intercept lna.

Suppose instead that you need to solve the equation $y = ka^x$. Taking natural logarithms gives $\quad lny = lnk + xlna$

Thus, if you plot lny against x the slope of the line would be lna and the intercept lnk.

NB. Any positive base can be used for the logarithms.