Integration Techniques

Integration by substitution

Suppose you wanted to evaluate the integral $\int_0^4 (2x-3)^4 dx$. One could expand $(2x-3)^4$ and integrate the resulting polynomial term by term. An alternative way is to consider the substitution $u = 2x - 3$. Differentiating gives $\frac{du}{dx} = 2$ or $dx = \frac{du}{2}$. Also, when $x = 0, u = -3$ and when $x = 4, u = 5$. The integral becomes

$$\int_0^4 (2x-3)^4 dx = \int_{-3}^5 u^4 \frac{du}{2} = \frac{1}{2}\left[\frac{u^5}{5}\right]_{-3}^5 = \frac{1}{10}\{5^5 - (-3)^5\} = \frac{3368}{10} = 336.8$$

Summary:-

(a) Choose an appropriate substitution to make the integrand easier to integrate.
(b) Determine the differential dx in terms of du.
(c) Change the limits to those appropriate to u.

Example: Perform the following indefinite integral. $\int \frac{f'(x)}{f(x)} dx$.

Here we make the substitution $u = f(x)$. Differentiating gives $\frac{du}{dx} = f'(x)$ or $du = f'(x)dx$.

Thus $\int \frac{f'(x)}{f(x)} dx = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$

NB. Since there are no limits to the integral it is convenient to express the answer in terms of the original variable x.

Integration by parts

We have already shown that integration is the reverse process to differentiation, or vice-versa. Consider the product rule in differentiation.

$$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right)v + u\left(\frac{dv}{dx}\right)$$

Now integrate throughout with respect to x to give

$\int \frac{d}{dx}(uv)dx = \int \left(\frac{du}{dx}\right) vdx + \int u\left(\frac{dv}{dx}\right) dx$ or more simply $uv = \int vdu + \int udv$

Rearranging gives $\int udv = uv - \int vdu$

This is the integration by parts formula. The only thing now is to choose u and v for the integral required.

Example: Find the integral $\int_0^1 xe^{-x} dx$.

We could define $u = x$ and $dv = e^{-x}, i.e.,$ by integrating $v = -e^{-x}$ to give

$$\int_0^1 xe^{-x}dx = [x(-e^{-x})]_0^1 - \int_0^1 (-e^{-x})dx = -e^{-1} + \int_0^1 e^{-x}dx$$
$$= -e^{-1} + [-e^{-x}]_0^1 = 1 - 2e^{-1}$$

Alternatively, we could define $u = e^{-x}$ and $dv = x, i.e. v = \frac{x^2}{2}$. This would give

$$\int_0^1 xe^{-x}dx = \left[(e^{-x})\frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{x^2}{2}(-e^{-x})dx = \frac{e^{-1}}{2} + \frac{1}{2}\int_0^1 x^2 e^{-x}dx$$

Whilst this last expression is correct it has not helped us to evaluate the original integral, in fact it has made it worse. Experience will guide you to the appropriate choice of u and v.

Example: The next example illustrates some cunning in defining u and v. Suppose we have the integral $\int (lnx)dx$. At first site it does not appear to be a product, but you can consider (lnx) to be $1 \times (lnx)$ and write the integral as $\int 1 \times (lnx)dx$.

Taking $u = lnx$ and $dv = 1, i.e., v = x,$ we can write

$$\int (lnx)dx = [xlnx] - \int x\left(\frac{dlnx}{dx}\right)dx = xlnx - \int x \times \frac{1}{x}dx = xlnx - \int dx = xlnx - x + C$$

Integration using partial fractions

The topic **Partial Fractions** is discussed in the algebra section. Essentially we can write

$$\frac{5(x^3-x^2-2x-4)}{(x^2-1)(x^2+4)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{(Cx+D)}{(x^2+4)}$$

We multiply throughout by $(x^2-1)(x^2+4)$ and compare coefficients of x, or evaluate both sides for specific values of x, to find the values of A, B, C and D. Here we find that

$$\frac{5(x^3-x^2-2x-4)}{(x^2-1)(x^2+4)} = \frac{2}{(x+1)} - \frac{3}{(x-1)} + \frac{(6x)}{(x^2+4)}$$

Integrating both sides gives

$$\int \frac{5(x^3-x^2-2x-4)}{(x^2-1)(x^2+4)}dx = \int \frac{2}{(x+1)}dx - \int \frac{3}{(x-1)} + \int \frac{(6x)}{(x^2+4)}dx$$

$$= 2\ln|x+1| - 3\ln|x-1| + 3ln|x^2+4| + C$$

Clearly this approach is only useful if the integrand has a denominator that can be factorised.

Note. There are other functions to consider for integration such as $sin^{-1}x$ but these do not appear in the single maths syllabus and so will be included later in other knowledge organisers when we consider the double maths syllabi.