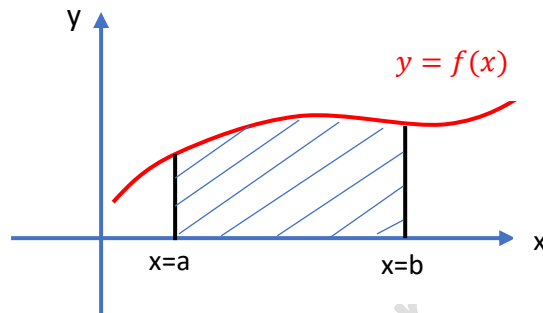


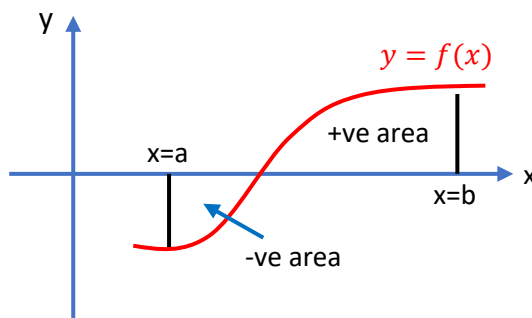
Integration, an introduction

Introduction

The integral of a function $f(x)$ over the x interval $[a,b]$, or between the limits $[a,b]$, is written $\int_{x=a}^{x=b} f(x)dx$ and is the area under the function $f(x)$ and the x -axis between $x = a$ and $x = b$. This is sketched alongside, with the area being shaded. Numerical methods for performing integration are considered in the knowledge organiser with that name. If the integral has limits, we refer to it as a **definite integral** and if not an **indefinite integral**. You will see the distinction later.



Care:- It should be noted that the integral gives the algebraic area, i.e., the sum of the positive area and negative area. If you want the actual area, disregarding the sign, then you have to perform two integrals, one where the function is positive and the other where the function is negative. Ignoring the sign of the integral and adding gives the absolute area. A good, more detailed introduction, can be found on the website Alevelmathematicstutor.com



Integrating polynomials

It can be shown that, for $n \neq -1$, the definite integral

$$\int_{x=a}^{x=b} x^n dx = \left[\frac{x^{n+1}}{(n+1)} \right]_a^b = \frac{1}{(n+1)} (b^{n+1} - a^{n+1}).$$

Note:- the square bracket contains the integrated function and the limits appear as a superscript and subscript.

A polynomial is a sum or difference of terms of the form x^n for some values of n , and we **integrate term by term**, eg.

$$\begin{aligned} \int_{x=a}^{x=b} (x^2 + 2x + 4) dx &= \int_{x=a}^{x=b} x^2 dx + 2 \int_{x=a}^{x=b} x dx + \int_{x=a}^{x=b} 4 dx \\ &= \left[\frac{x^3}{3} \right]_a^b + 2 \left[\frac{x^2}{2} \right]_a^b + 4 \left[\frac{x^1}{1} \right]_a^b \end{aligned}$$

Note $\int_{x=a}^{x=b} 4 dx = 4 \int_{x=a}^{x=b} x^0 dx = 4 \left[\frac{x^1}{1} \right]_a^b$

For an indefinite integral, we integrate the function as before, but without the limits, and add a constant to the integrated function, i.e.,

$$\int x^n dx = \frac{x^{n+1}}{(n+1)} + C, \quad \text{where } C \text{ is referred to as the constant of integration.}$$

This does not change the definite integral because

$$\int_{x=a}^{x=b} x^n dx = \left[\frac{x^{n+1}}{(n+1)} + C \right]_a^b = \left(\frac{b^{n+1}}{(n+1)} + C \right) - \left(\frac{a^{n+1}}{(n+1)} + C \right) = \frac{b^{n+1}}{(n+1)} - \frac{a^{n+1}}{(n+1)}, \text{ the same result.}$$

Fundamental Theorem of Calculus

Suppose $F(x) = \int_{-\infty}^x g(s) ds$

(Note the bottom limit would change if the domain of $g(y)$ had a finite lower limit.)

$$F(x+h) - F(x) = \int_{-\infty}^{x+h} g(s) ds - \int_{-\infty}^x g(s) ds = \int_x^{x+h} g(s) ds$$

If h is small, the area under the curve $y = g(s)$ between the limits $[x, x+h]$ would be approximately the height $g(x)$ times the width h , the approximation becoming exact as h tends to zero. Thus, we can write

$$F(x+h) - F(x) \approx g(x)h \quad \text{or} \quad \frac{F(x+h) - F(x)}{h} \approx g(x)$$

Taking the limit $h \rightarrow 0$ on both sides gives $\frac{dF(x)}{dx} = g(x)$, i.e., $\frac{d}{dx} \left(\int_{-\infty}^x g(s) ds \right) = g(x)$

This is the fundamental theorem of calculus and indicates that integration is the reverse process to differentiation and vice-versa **to within an additive constant term.**

Note. $\frac{d}{dx} (x^2 + x + 4) = 2x + 1$ but $\int (2x + 1) dx = x^2 + x + C$. From the integral C is unknown, but if you are told the function is 4 when $x = 0$ then this identifies C as 4.

Example: We have seen that $\int_{-\infty}^x s^n ds = \frac{x^{n+1}}{(n+1)} + C$, where C is an integration constant.

Differentiating $\frac{x^{n+1}}{(n+1)} + C$ with respect to x gives $\frac{1}{(n+1)} ((n+1)x^n) = x^n$.

The consequences of the Fundamental Theorem of calculus are very important. For example we know that $\frac{d}{dx} \sin(ax) = a \cos(ax)$ therefore we can write

$$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + C.$$

Also $\frac{d}{dx} e^{kx} = k e^{kx}$. Hence $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$. A table of standard integrals is shown below. Note, these are indefinite integrals and so should have a constant added when used.

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$x^n \quad n \neq -1$	$\frac{x^{n+1}}{(n+1)}$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right) \quad x < a$
$\frac{1}{x}$	$\ln x $	$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
$\sin(x)$	$-\cos(x)$	$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}(x) = \ln x + \sqrt{x^2 - a^2} $ for $ x > a $
$\cos(x)$	$\sin(x)$	$\frac{1}{\sqrt{x^2 - a^2}}$	$\sinh^{-1}(x) = \ln x + \sqrt{x^2 + a^2} $
$\tan(x)$	$\ln \sec(x) $	$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right)$
$\cot(x)$	$\ln \sin(x) $	$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $
$\operatorname{cosec}(x)$	$-\ln \operatorname{cosec}(x) + \cot(x) $	$\sec^2 x$	$\tan x$
$\sec(x)$	$\ln \sec(x) + \tan(x) $		

Areas between curves

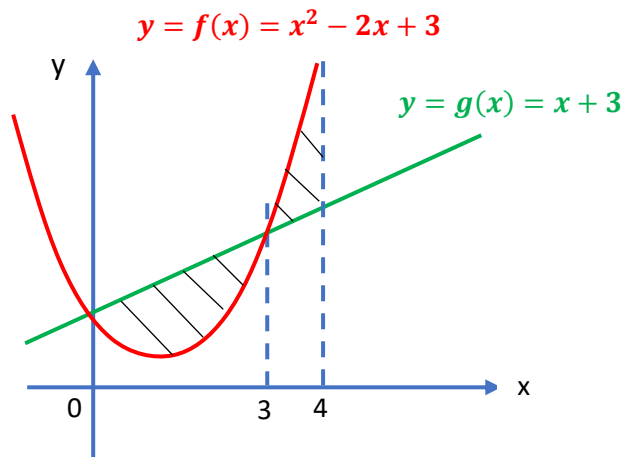
Suppose you want to identify the area between the two functions $y = f(x) = x^2 - 2x + 3$ and $y = g(x) = x + 3$ when $0 \leq x \leq 4$ (shaded area in the graph). A sketch of these functions is shown alongside. A number of integrals need to be performed, and are shown below.

$$\int_0^3 (x^2 - 2x + 3) dx = \left[\frac{x^3}{3} - x^2 + 3x \right]_0^3 = 9$$

$$\int_0^4 (x^2 - 2x + 3) dx = \left[\frac{x^3}{3} - x^2 + 3x \right]_0^4 = \frac{52}{3}$$

$$\int_0^3 (x + 3) dx = \left[\frac{x^2}{2} + 3x \right]_0^3 = \frac{27}{2}$$

$$\int_0^4 (x + 3) dx = \left[\frac{x^2}{2} + 3x \right]_0^4 = 20$$



The required area is the difference of two components.

Firstly, the area under the line, but above the quadratic when

$$0 \leq x \leq 3 \quad \text{is} \quad \frac{27}{2} - 9 = \frac{9}{2}$$

Secondly, the area under the quadratic and above the line when

$$3 \leq x \leq 4 \quad \text{is} \quad \left(\frac{52}{3} - 9 \right) - \left(20 - \frac{27}{2} \right) = \frac{11}{6}.$$

The **TOTAL** shaded area = $\frac{9}{2} + \frac{11}{6} = \frac{38}{6} = \frac{19}{3}$.

If one ignored the cross point and evaluated the area under each curve between 0 and 4 the answer would have been $\frac{52}{3} - 20 = -\frac{8}{3}$ or $\frac{8}{3}$, depending on the order of subtraction.