Functions4 More technical features of functions

Notation

So far, we have used the word function in a general sense, i.e., quadratic function, sine function, log function, etc. Hopefully there has been no confusion. Now we need to investigate the technical features of functions. To highlight these features a variety of terminologies have arisen. In particular, some would prefer to use mappings, or transformations or functions to describe these features. The main thing is that the reader understands the points and ideas that are being explained. The main two descriptions are functions and mappings. Some examination boards prefer mappings and so, in this relatively short section we will emphasize both at the key points. Check with your examination board which notation is preferred.

More technical details about functions

Domain of a function: - The domain of a function $f(x)$ is the set of values of x for which the function is defined.

Range of a function: - The range of a function $f(x)$, defined on a domain $\mathcal{D}(f(x))$, is the set of values derived from $f(x)$ as x goes through all possible values in the domain and denoted by $\mathcal{R}(f(x))$.

Example: Consider the function $f(x) = \sqrt{x-1}$.

The maximum domain would be $\mathcal{D}(f(x)) = [1,\infty)$, all subsets are valid domains.

The maximum range of $f(x) = \mathcal{R}(f(x)) = [0,\infty)$.

If one wanted to work with a domain of $[5,17]$ the corresponding range would be $[2,4]$.

Care:- If $f(x) = x^2 - 1$ with domain $\mathcal{D}(f(x)) = (-1,2]$, the range is [-1,3], i.e., you cannot necessarily deduce the range by evaluating the function at the endpoints of its domain. $f(x)$ has a minimum value of -1 when $x = 0$, and since 0 is within the domain, - 1 is within the range. Draw a sketch to be sure.

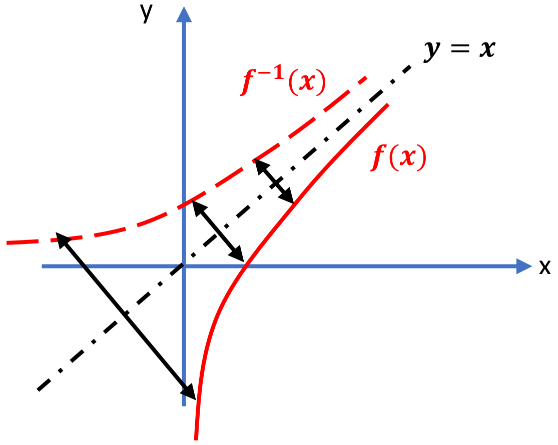
One to many mappings(functions): - Suppose the mapping(function) $f(x) = \pm\sqrt{x}$, with domain [0,∞)
For each value of x there are 2 values of y. The fact that there is more than one y value for each x value means this mapping(function) is a **one to many mapping(function) of x**.

Many to one mapping(function): - Suppose the function $f(x) = \sin(x)$, with domain $(-\infty,\infty)$. Then when $x = \pm n\pi \quad n = 0,1,2, \ldots.$ $\sin(x) = 0$, i.e., there are many x values that gives rise to the same function value. We call this
a many to one mapping(function).

Many to many mapping(function). Suppose we consider the equation of a circle in the form $x^2 + y^2 = r^2$, with r known. When x is specified there are two values of y and when y is specified there are two values of x.
This would be an example of a **many to many mapping(function).**

One to One function:- The name defines it, in that for any one value of x there is only one value of y, and for any one value of y there is only one corresponding value of x. This is a very important type of function because if we go from $x \to y$ by the function $f(x)$ we know we can return from $y \to x$ by a function $g(y)$ say. Using the terminology $g(y)$ does not make the process of returning to x obvious and so it is convention to use the notation $f^{-1}(y)$ the superscript (-1) indicating that you are undoing what the function $f(x)$ has done. The variable names do not matter and so we say $f^{-1}(x)$ is the inverse function for $f(x)$. Notationally we can write $f^{-1}(f(x)) = x$, i.e., $f(x) \to y$ and $f^{-1}(y) \to x$. From the very explanation above it is clear that a **one to one function has an inverse**.

NB It can be shown that the graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$, i.e., the $45°$ line through the origin as shown in the figure.



Restricting the domain

At first sight the above conditions would seem to exclude say, $sinx$ from having an inverse because it is not a one-to-one function. You need to remember that the domain is very much part of defining a mapping(function). If we consider $sinx$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ it is a one-to-one function with inverse $sin^{-1}x$. This is what is used in calculators etc. We then work with the acronym "CAST", or an equivalent acronym, to determine both values in the range $[-\pi, \pi]$ and consider adding/subtracting appropriate multiples of 2π to determine all solutions. This process can also be achieved by using suitably clear graphs.

Finding an inverse function

The function $f(x) = \frac{2+x}{1+x}$ is defined on the domain $(-1, \infty)$. Determine the inverse function $f^{-1}(x)$ and specify its domain and range.

We can rewrite $f(x)$ as $\quad y = f(x) = \frac{(1+x)+1}{(1+x)} = 1 + \frac{1}{(1+x)}$.

In this form, it follows that as $x \to (-1)$, $f(x) \to \infty$, and that as $x \to \infty$, $f(x) \to 1$ from above. It therefore follows that the range of $\mathcal{R}(f(x)) = (1, \infty)$. Now, solving for x in the above equation gives $\quad (y-1) = \frac{1}{(1+x)} \quad$ or $\quad (1+x) = \frac{1}{(y-1)} \quad i.e., \quad x = \frac{1}{(y-1)} - 1 = \frac{2-y}{y-1}$

We know that $f^{-1}(y) = x$ and so $f^{-1}(y) = \frac{2-y}{y-1}$. In terms of x this would be written as $f^{-1}(x) = \frac{2-x}{x-1}$. (**NB** the variable used does not matter it is the rule that matters.) Because of the one-to-one nature of this function we can write

$$\mathcal{D}(f^{-1}(x)) = \mathcal{R}(f(x)) = (1, \infty) \quad \text{and} \quad \mathcal{R}(f^{-1}(x)) = \mathcal{D}(f(x)) = (-1, \infty)$$

Composite functions

The above notation of a sequence of two functions, $f^{-1}(f(x))$ can be extended to any two functions as follows. Suppose we have two functions $f(x)$ and $g(x)$, with domains $\mathcal{D}\big(f(x)\big)$ and $\mathcal{D}(g(x))$ and ranges $\mathcal{R}\big(f(x)\big)$ and $\mathcal{R}(g(x))$ respectively. The composite function $fg(x)$ is defined to be $f(g(x))$, i.e., $g(x)$ is formed first and then $f(g(x))$, which means wherever you have x in the formula defining $f(x)$ you put $g(x)$ instead.

NB. For f to operate on g we require the range of $g(x)$ to be contained within the domain of $f(x)$.

Example: Define the functions $f(x)$and $g(x)$ as

$$f(x) = x^2 + 2x - 1 \qquad \text{with domain } (-\infty, \infty)$$

$$g(x) = \sqrt{x^2 - 1} \qquad \text{with domain } [1, \infty)$$

We note the range of $g(x)$ is $[0, \infty)$, which is within the domain of $f(x)$ and so

$$f\big(g(x)\big) = (g(x))^2 + 2g(x) - 1 = (x^2 - 1) + 2\sqrt{x^2 - 1} - 1$$

The range of $f\big(g(x)\big)$ is the image of $[0, \infty)$ under $f(x)$ which is $[-1, \infty)$.