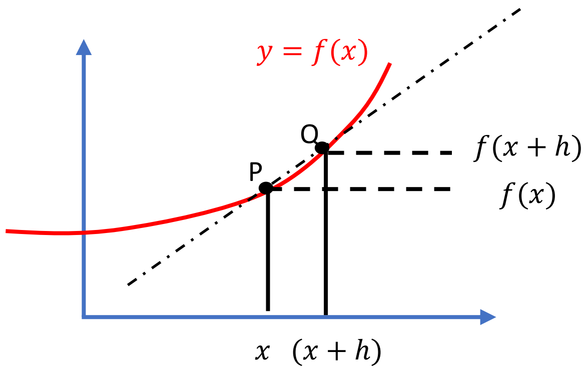
Differentiation1:-Basic Rules & classification of turning points

Differentiation

Consider the figure showing a plot of $y = f(x)$ and two points P and Q on the curve with x values of x and $(x+h)$ respectively. PQ is a chord of the graph with slope

$$\text{Slope of chord} = \frac{f(x+h)-f(x)}{h}$$



As Q moves towards P the chord becomes the tangent at P. This prompts the following definition.

Slope of the tangent at P $= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and is denoted by $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$ or $f'(x)$. These are referred to as the **derivative** of the function $f(x)$.

Note. If you are asked to find the derivative of a function from first principles then you have to evaluate this limit directly. Below is a list of common derivatives.

$f(x)$	$\frac{df(x)}{dx}$	$f(x)$	$\frac{df(x)}{dx}$
$x^n \quad n \neq 0$	nx^{n-1}	$\sinh(x)$	$\cosh(x)$
$\sin(x)$	$\cos(x)$	$\cosh(x)$	$\sinh(x)$
$\cos(x)$	$-\sin(x)$	$\tanh(x)$	$sech^2(x)$
$\tan(x)$	$sec^2(x)$	$sinh^{-1}(x)$	$\frac{1}{\sqrt{1+x^2}}$
$\sec(x)$	$\sec(x)\tan(x)$	$cosh^{-1}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\cot(x)$	$-cosec^2(x)$	$tanh^{-1}(x)$	$\frac{1}{1-x^2}$
$cosec(x)$	$-cosec(x)\cot(x)$	$\ln(x)$	$\frac{1}{x}$
$sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$	e^x	e^x
$cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$e^{f(x)}$	$f'(x)e^{f(x)}$ **Chain Rule**
$tan^{-1}(x)$	$\frac{1}{1+x^2}$	$lnf(x)$	$\frac{f'(x)}{f(x)}$ **Chain Rule**

Useful rules for differentiation

Product Rule

Suppose the function $f(x)$ can be written as the product $f(x) = u(x)v(x)$ then the product rule gives

$$\frac{df(x)}{dx} = \frac{d}{dx}(u(x)v(x)) = \left\{\frac{du(x)}{dx}\right\} \times v(x) + u(x) \times \left\{\frac{dv(x)}{dx}\right\}$$

i.e., you move the derivative through term by term leaving the remain functions untouched.

Quotient Rule

Suppose the function $f(x)$ can be written as the ratio $f(x) = \frac{u(x)}{v(x)}$ then the quotient rule gives

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{\left(\frac{du(x)}{dx}\right)v(x) - u(x)\left(\frac{dv(x)}{dx}\right)}{(v(x))^2}$$

Chain Rule

Suppose $f(x)$ can conveniently be written as a function of u, where u is expressed as a function of $x, i.e., u(x)$.

For example, we could write $f(x) = (2x+4)^4$ as $f(u) = u^4$ where $u = 2x + 4$.

In this situation $\frac{df}{dx} = \left(\frac{df}{du}\right) \times \left(\frac{du}{dx}\right)$ ← **This is the chain rule**

In the above example $\frac{df}{du} = \frac{d}{du}(u^4) = 4u^3$ and $\frac{du}{dx} = \frac{d}{dx}(2x+4) = 2$

Thus $\frac{df}{dx} = \left(\frac{df}{du}\right) \times \left(\frac{du}{dx}\right) = (4u^3)(2) = 8(2x+4)^3$

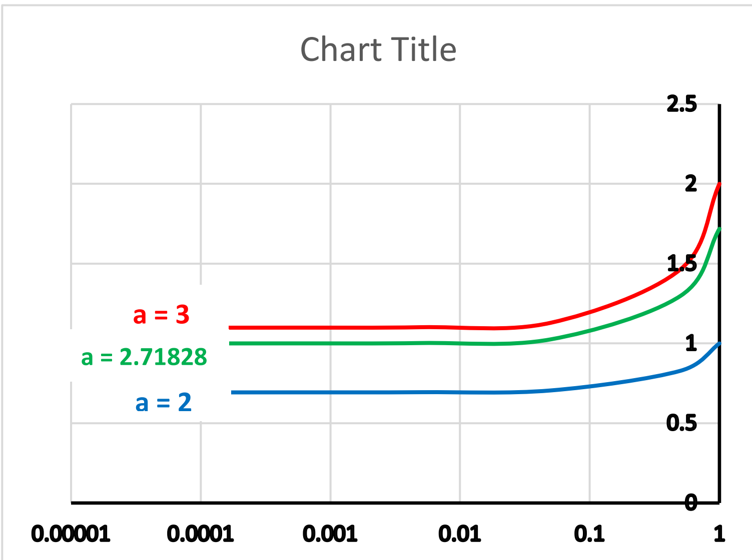
The exponential function e^x

Here we consider more generally the derivative of a^x. From first principles we can write

$$\frac{d}{dx}(a^x) = \lim_{h \to 0}\left(\frac{a^{x+h} - a^x}{h}\right)$$

$$= a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h}\right)$$

The limit $\lim_{h \to 0} \frac{a^h - 1}{h}$ is shown plotted for a variety of values for a. The limit is only 1 if the base a is taken to be 2.71828, which is the value of e.



Thus, we have deduced that $\frac{d}{dx}e^x = e^x$. Using the chain rule we can conclude that

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

We know that a^x can be written as $a^x = e^{xlna}$ and so

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{xlna}) = \ln(a) \times (e^{xlna}) = lna \times a^x \quad \text{or} \quad a^x lna$$

This explains the red and blue limits shown in the graph, $ln2 = 0.693$ and $ln3 = 1.098$.

Note. If $f(x) = kx$ then $\frac{d}{dx}e^{kx} = ke^{kx}$, i.e., the derivative of $y = e^{kx}$ is ky, which means the rate of change of y is proportional to y, a common assumption about rates of change of physical processes.

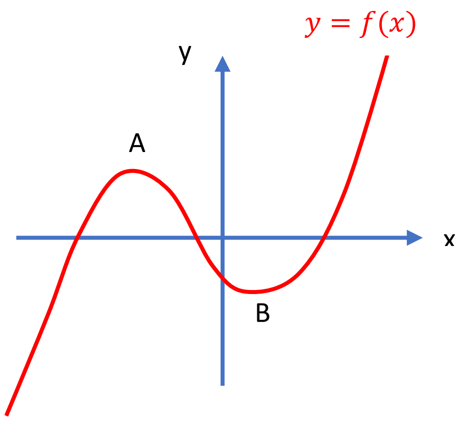
Also, if $y = lnx$ we can write $x = e^y$. Using the chain rule gives, by differentiating with respect to x throughout, $\frac{d}{dx}(x) = \frac{d}{dx}(e^y) = \frac{d}{dy}(e^y) \times \frac{dy}{dx} = e^y\frac{dy}{dx} = x\frac{dy}{dx}$

But $\frac{d}{dx}(x) = 1$, and so we can write $x\frac{dy}{dx} = 1$ or $\frac{dy}{dx} = \frac{1}{x}$ i.e., $\frac{d}{dx}(lnx) = \frac{1}{x}$.

We can use the Chain Rule again to write $\frac{d}{dx}lnf(x) = \frac{d}{du}lnu \times \frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{f'(x)}{f(x)}$, where the substitution $u = f(x)$ has been made.

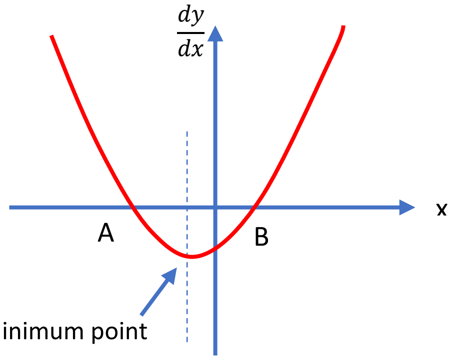
Maxima, minima and points of inflection.

Consider the sketch of the cubic shown. If there was no restriction on the x values taken, the maximum value would be $+\infty$ and the minimum value would be $-\infty$.



There are, however, **local maxima and minima** at A and B respectively. These are characterised by being at points where the curve turns, and as such are often referred to as **turning points**. Mathematically we would say that these points are where $\frac{dy}{dx} = 0$. Thus, solving this equation will give enable the coordinates of these turning points to be found. We can see that A is a local maximum and B is a local minimum, but how can we deduce this mathematically? For the maximum point A, we see that the gradient of the curve is positive to its left and negative to its right, whereas for the local minimum at B the reverse is true.

A sketch of $\frac{dy}{dx}$ is shown. Moving from the far left of the graph of $f(x)$ we see that $\frac{dy}{dx}$ starts at a large positive value, reducing to 0 at A and continuing to decrease to a point between A and B from which it starts to increase again through 0 at B and increasingly positive there afterwards.



The gradient of this curve is $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. The gradient is seen to be negative at A and positive at B. This is the condition we are looking for to classify A and B as maxima or minima. The minimum point on this graph $\left(\frac{d^2y}{dx^2} = 0\right)$ is called the **point of inflection** and is where the **curvature of the curve changes**. On the left the cubic curves downwards and on the right it curves upwards. **This last point is crucial for it to be a valid point of inflection**. This will be true if $\frac{d^2y}{dx^2}$ **changes sign at this point.**

In summary:-

(i) $\frac{df(x)}{dx} = 0$ determines the turning points

(ii) At the turning points $\frac{d^2y}{dx^2} < 0$ **for a maximum** and $\frac{d^2y}{dx^2} > 0$ **for a minimum.**

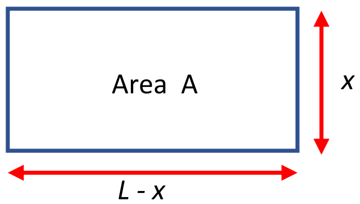
(iii) A **point of inflection is where** $\frac{d^2y}{dx^2} = 0$**, and** $\frac{d^2y}{dx^2}$ **changes sign as well.**

Example: Consider the function $y = x^n$, $n = 2,3, \ldots$ .... When is $x = 0$ a point of inflection? Differentiating twice gives $\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$. We note that $\frac{d^2y}{dx^2} = 0$ for all n = 2,3,...

When $(n-2)$ is even $\frac{d^2y}{dx^2} = n(n-1)x^{n-2} \geq 0$ for all x and so $x = 0$ is not a point of inflection. If however, $(n-2)$ is odd, then $\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$ changes sign at $x = 0$ and so this point is a point of inflection. In summary $x^3, x^5, x^7, \ldots$.. have a point of inflection at $x = 0$, whereas $x^2, x^4, x^6, \ldots$.. do not.

Example:- A string of length L cm is folded at a point x cm from one end to form two adjacent sides of a rectangle. What is the value of x that gives the maximum area?

An illustration of the situation is shown alongside.



The area formed, in terms of x, is

$A = x(L-x) = xL - x^2$

Differentiating gives $\frac{dA}{dx} = L - 2x$

The turning point satisfies $\frac{dA}{dx} = 0$, i.e., $L - 2x = 0 \Rightarrow x = \frac{L}{2}$ cm.

Differentiating again gives $\frac{d^2A}{dx^2} = -2 < 0$, therefore the turning point is a maximum and the maximum area is $\frac{L^2}{4}$.