Coordinate Geometry 2:- Parametric equations

The equation of a line or circle has been defined by the type of relation between the variables x and y. Sometimes it proves convenient to define both x and y through another variable t, say. Consider the parametric equations

$x = 5cost$ and $y = 5sint$, where the parameter t can take all values in the range $[0,2\pi]$.

Squaring and adding gives $x^2 + y^2 = 25cos^2t + 25sin^2t = 25(cos^2t + sin^2t) = 25$

This can be recognised as the equation of a circle with a radius of 5 and with the origin as its centre.

NB If t was restricted to the values $[0,\pi]$ then the equation would be that of a semicircle. You must consider the range of the parameter before concluding the type of shape being created.

Example: The parametric form for the equation of a circle with centre (f,g) and radius r is

$$x = f + rcost \quad \text{and} \quad y = g + rsint$$

There are other common forms, for example $x = 5t$ and $y = \frac{5}{t}$ for $t \in (0,\infty)$. All we have to do is to eliminate t to find the relationship between x and y. From the second equation $t = \frac{5}{y}$. Substituting into the first equation gives $x = 5\left(\frac{5}{y}\right)$ or $xy = 25$.

We note the range of x and y is also $(0,\infty)$. **This curve is referred to as a rectangular hyperbola.** In a similar way we can write

Ellipse:- The parametric form is $(x,y) = (acos\theta, bsin\theta)$ for $\theta \in [0°, 360°)$

giving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parabola:- The parametric form is $(x,y) = (at^2, 2at)$ for $t \in (-\infty,\infty)$

giving $y^2 = 4ax$

Hyperbola:- The parametric form is $(x,y) = (asec\theta, btan\theta)$ for $\theta \in (-90°, 90°)$

giving $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Finding the gradient of the curve at a point

This is essential when having to find the equation of the tangent or normal to a curve at a point on the curve. It can be done in two ways, implicit differentiation if the curve is given in (x,y) form, or using the chain rule for differentiation if given in parametric form. This is now done for the curve types above.

Rectangular hyperbola

The equation is $xy = c^2$ with parametric form $x = ct$ and $y = \frac{c}{t}$ for $t \in (0,\infty)$

Differentiating implicitly gives $\frac{d}{dx}(xy) = \frac{d}{dx}(c^2)$ $i.e.,$ $y + x\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{y}{x}$.

Using the Chain Rule: $\frac{dx}{dt} = c$ and $\frac{dy}{dt} = -\frac{c}{t^2}$.

Hence $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \div \left(\frac{dx}{dt}\right) = -\left(\frac{c}{t^2}\right) \div (c) = -\frac{1}{t^2} = -\frac{y}{x}$. Both approaches give the same answer, the problem will decide which is the most appropriate.

Ellipse

The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with parametric form $x = acos\theta$ and $y = bsin\theta$ for $\theta \in [0,360^\circ)$.

Differentiating implicitly gives $\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$ $i.e.,$ $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{xb^2}{ya^2}$.

Using the Chain Rule: $\frac{dx}{d\theta} = -asin\theta$ and $\frac{dy}{d\theta} = bcos\theta$.

Hence $\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) \div \left(\frac{dx}{d\theta}\right) = (bcos\theta) \div (-asin\theta) = -\frac{bcos\theta}{asin\theta} = -\frac{b^2x}{a^2y}$. Both approaches give the same answer, the problem will decide which is the most appropriate.

Parabola

The equation is $y^2 = 4ax$ with parametric form $x = at^2$ and $y = 2at$ for $t \in (-\infty, +\infty)$.

Differentiating implicitly gives $\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$ $i.e.,$ $2y\frac{dy}{dx} = 4a$ or $\frac{dy}{dx} = \frac{2a}{y}$.

Using the Chain Rule: $\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$.

Hence $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \div \left(\frac{dx}{dt}\right) = (2a) \div (2at) = \frac{1}{t} = \frac{2a}{y}$. Both approaches give the same answer, the problem will decide which is the most appropriate.

Hyperbola

The equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with parametric form $x = asec\theta$ and $y = btan\theta$ for $\theta \in (-90^\circ, 90^\circ)$.

Differentiating implicitly gives $\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$ $i.e.,$ $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \frac{xb^2}{ya^2}$.

Using the Chain Rule: $\frac{dx}{d\theta} = asec\theta tan\theta$ and $\frac{dy}{d\theta} = bsec^2\theta$.

Hence $\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) \div \left(\frac{dx}{d\theta}\right) = (bsec^2\theta) \div (asec\theta tan\theta) = \frac{bsec\theta}{atan\theta} = \frac{b}{a}cosec\theta = \frac{b\left(\frac{x}{a}\right)}{a\left(\frac{y}{b}\right)} = \frac{b^2x}{a^2y}$.

Both approaches give the same answer, the problem will decide which is the most appropriate.

NB It should be remembered that parametric forms can be used as substitutions when integrating.