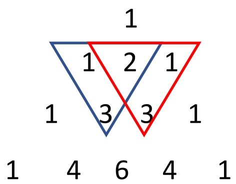
Binomial Expansion

Here we want to expand $(x+a)^n$ without multiplying all of the factors at length. Below you can see the expansions for n = 2, 3 and 4. We note that for $(x+a)^2$ all terms have total power 2 when you include both powers of a and x. For $(x+a)^3$ all terms have total power 3 and so on. The coefficients of the terms are shown in the number triangle alongside. We note that the 3 in the bottom apex of the first triangle is obtained by adding the numbers in the top two vertices, in this case 1 + 2. Following this procedure, as the triangle moves across the numbers we obtain 1,3,3,1 from 1,2,1. After obtaining 1,3,3,1, following the same procedure we obtain 1,4,6,4,1, which are the coefficients for the expansion of $(x+a)^4$. This method continues for all integer powers n, and the method is referred to as **PASCAL'S TRIANGLE**. The main drawback is that you need the full triangle, a real problem if you want to expand $(x+a)^{10}$ say.

$$(x+a)^2 = x^2 + 2ax + a^2$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$



Using combinations

The number of combinations of selecting r objects from n is $C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$. In the case $n = 3$ and $r = 1$ we obtain $C_1^3 = \binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{3!}{1!2!} = \frac{3\times2\times1}{1\times2\times1} = 3$. This is the second coefficient in $(x+a)^3$.

Generally

$(x+a)^n = \sum_{r=0}^{n} C_r^n a^r x^{n-r} \equiv \sum_{r=0}^{n} C_r^n x^r a^{n-r}$ (You can expand with a leading or x leading)

This symmetry arises because $C_{n-r}^n = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = C_r^n$.

We also note that the ratio of two successive coefficients is

$(C_{r+1}^n) \div (C_r^n) = \frac{n!}{(r+1)(n-r-1)!} \times \frac{r!(n-r)!}{n!} = \frac{(n-r)}{(r+1)}$, the first coefficient being $C_0^n = \frac{n!}{0!n!} = 1$

The sequence of coefficients becomes $1, \ \frac{(n-0)}{(0+1)} = (n), \ \frac{(n-1)}{(1+1)} \times (n) = \frac{(n)(n-1)}{2}, \ldots$

i.e., $\quad (x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}a^3 + \cdots$..

This enables us to write a series expansion for functions of the form $(1+x)^{-1}$, i.e.,

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \ldots$$

Because this is an infinite series we need to be careful about convergence. It can be shown to converge if $|x| < 1$. Changing the sign of x in the above gives

$$(1-x)^{-1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots \quad \text{for } |x| < 1$$

Generally $\quad (a+bx)^{-1} = \frac{1}{a}\left(1 + \frac{bx}{a}\right)^{-1} = \frac{1}{a}\left\{1 - \left(\frac{bx}{a}\right) + \frac{1}{2!}\left(\frac{bx}{a}\right)^2 - \frac{1}{3!}\left(\frac{bx}{a}\right)^3 \ldots\right\} \quad$ for $\left|\frac{bx}{a}\right| < 1$

The above also works for rational n, i.e., $n = \frac{1}{2}, \frac{2}{3}, \ etc$

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