Partial fractions

Here we want to write $\frac{S(x)}{f_1(x)f_2(x)f_3(x)}$ in the form $h(x)+\frac{g_1(x)}{f_1(x)}+\frac{g_2(x)}{f_2(x)}+\frac{g_3(x)}{f_3(x)}$

If the degree of $S(x)$ is greater than or equal to the degree of $f_1(x)f_2(x)f_3(x)$ then we can use long division (**see Algebra revision sheet on division of polynomials**) to write

$$S(x)=Q(x)\times\big(f_1(x)f_2(x)f_3(x)\big)+R(x),$$

where $Q(x)$ is the quotient and $R(x)$ the remainder.

As a result $\frac{S(x)}{f_1(x)f_2(x)f_3(x)}=Q(x)+\frac{R(x)}{f_1(x)f_2(x)f_3(x)}$ and therefore we take $h(x)=Q(x)$

The problem reduces to one of writing $\frac{R(x)}{f_1(x)f_2(x)f_3(x)}=\frac{g_1(x)}{f_1(x)}+\frac{g_2(x)}{f_2(x)}+\frac{g_3(x)}{f_3(x)}$ where the degree of $R(x)$ is less than that of $f_1(x)f_2(x)f_3(x)$, which means

$$Degree\big(g_i(x)\big)<Degree\big(f_i(x)\big)\quad \text{for } i=1,2,3$$

NB. Currently, for UK examination Boards, the degree of each $f_i(x)$ is at most 2, i.e., they only consider linear or quadratic factors.

If $f_i(x)$ is linear, i.e., (a_ix-b_i) then the degree of $f_i(x)$ is 1 and therefore the degree of $g(x)$ is 0, i.e., it is a constant.

We therefore write $\frac{g_i(x)}{f_i(x)}=\frac{A}{(a_ix+b_i)}$, where A is a constant to be determined.

If $f_i(x)$ is quadratic, i.e., $(a_ix^2+b_ix+c_i)$ then the degree of $f_i(x)$ is 2 and therefore the degree of $g(x)$ is 1, i.e., it is a linear function.

We therefore write $\frac{g_i(x)}{f_i(x)}=\frac{Ax+B}{(a_ix^2+b_ix+c_i)}$, where A and B are constants to be determined.

NB. A factor of $(ax+b)^2$ can be written in any one of the following two equivalent forms

$$\frac{Ax+B}{(ax+b)^2}\quad\text{or}\quad\frac{C}{(ax+b)}+\frac{D}{(ax+b)^2}\ \left\{=\frac{C(ax+b)+D}{(ax+b)^2}=\frac{Ex+F}{(ax+b)^2}\right\}$$

As can be seen the two forms are equivalent.

Example: Express $\frac{(-x^2+5x+5)}{(x-1)(x+2)^2}$ in partial fraction form.

Here we write $\frac{(-x^2+5x+5)}{(x-1)(x+2)^2}\equiv\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$.

Multiplying throughout by $(x-1)(x+2)^2$ gives

$$(-x^2+5x+5)\equiv A(x+2)^2+B(x-1)(x+2)+C(x-1)$$

The above is an identity, i.e., its valid for all x, and so we can equate the coefficients of the same powers of x, or we can evaluate both sides for a specified x, or a mixture of both.

Setting $x=1$ gives $-1+5+5=9A\quad i.e.,\quad 9=9A\quad\text{or}\quad A=1$

Setting $x=-2$ gives $-4-10+5=-3C\quad i.e.,\quad -9=-3C\quad\text{or}\ C=3$

Equating coefficients of x^2 gives $-1=A+B\quad i.e.,\ B=-1-A=-1-1=-2$

Thus we may write $\frac{(-x^2+5x+5)}{(x-1)(x+2)^2}\equiv\frac{1}{(x-1)}-\frac{2}{(x+2)}+\frac{3}{(x+2)^2}$