Division of Polynomials & the Factor Theorem

Division of polynomials

A polynomial is an expression of the form $3x^7 - 4x^5 + 2x^2 - 1$, i.e., it contains terms that are whole number powers of the variable x. The maximum power of x is called the **degree** of the polynomial. In the above example the degree is 7. Long division of polynomials is very similar to long division of numbers.

Example.

Suppose we wish to evaluate the following $\frac{(6x^3+11x^2-11x+2)}{(2x-1)}$. We would do it as follows

	$3x^2 + 7x - 2$	Quotient
Divisor $(2x-1)$	$6x^3 + 11x^2 - 11x + 2$	
	$3x^3 - 3x^2$	Multiply divisor by $3x^2$ and subtract
	$14x^2 - 11x$	
	$14x^2 - 7x$	Multiply divisor by $7x$ and subtract
	$-4x + 2$	
	$-4x + 2$	Multiply divisor by -2 and subtract
	0	No remainder

Since there is no remainder, we can write $6x^3 + 11x^2 - 11x + 2 \equiv (2x-1)(3x^2 + 7x - 2)$.

Generally, if we want to evaluate $\frac{f(x)}{h(x)}$, we can write $f(x) \equiv h(x)Q(x) + R(x)$, where $Q(x)$ is the quotient and $R(x)$ is the remainder, which is a **polynomial with degree less than the divisor.**

Remainder Theorem – Factor Theorem

Let us think of dividing $f(x)$ by the linear factor $(x-a)$. From above we can write

$$f(x) = (x-a)Q(x) + R(x),$$

where the degree of R is less than 1, i.e., it is of degree 0, which means it is a constant.

It follows that $f(a) = (a-a)Q(a) + R = 0 + R = R$. If $f(a) = 0$ then $R = 0$ and so we can write

$$f(x) \equiv (x-a)Q(x) \quad \text{i.e.,} \quad (x-a) \text{ is a factor of } f(x).$$

Generally if $f\left(\frac{a}{b}\right) = 0$ we can conclude that $\left(x - \frac{a}{b}\right)$ or $(bx - a)$ is a factor of $f(x)$. After the factor has been found the quotient $Q(x)$ can be determined by inspection or by long division.