Indices and Surds

Indices

This is really a shorthand way of writing products of the form $a \times a \times a \times a \times a$ etc.

If there are n factors, we write $a^n = a \times a \times a \times \ldots\ldots \times a \times a \quad (n \text{ factors})$ and n is referred to as the index, or power of a.

Using this interpretation, we have the following rules of indices

(i) $a^{n+m} = a^n \times a^m$ There are $n + m$ factors of a both sides.

(ii) $a^{n-m} = \frac{a^n}{a^m}$ m factors of a cancel, leaving $(n - m)$ factors of a.

(iii) $a^0 = \frac{a^n}{a^n} = 1$ The n factors in the numerator and denominator cancel

(iv) $a^{-m} = \frac{1}{a^m}$ Take $n = 0$ in (ii) and use (iii).

(v) $(a^n)^m = a^{nm}$ There are nm factors of a both sides.

The above results have been derived thinking of n and m as integers. They do extend to when they are rational. It can be shown that we can write.

$$\sqrt{a} = a^{\frac{1}{2}} \qquad \sqrt[3]{a} = a^{\frac{1}{3}} \qquad \sqrt[n]{a} = a^{\frac{1}{n}} \qquad \sqrt[n]{a^m} = a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \qquad a^{-\frac{n}{m}} = \frac{1}{a^{\frac{n}{m}}}$$

Surds

These are expressions that involve use of a root symbol, that is square root, cube root or higher roots. The advantage of developing a notation for various roots is that it can be written down precisely, whereas its decimal form would have an infinity of places. Eg,

$\sqrt{2} = 1.414213562\ldots$ to 9 decimal places from a calculator

The rules for surds will work for any numbers and so there is no problem with using the rules generally.
If x and y are two positive numbers, then the following rules apply

(i) $\left(\sqrt{x}\right)^2 = x \qquad \left(\sqrt[n]{x}\right)^n = x$

(ii) $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. For any a and b $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$

(iii) $\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right) = \left(\sqrt{x}\right)^2 - \left(\sqrt{y}\right)^2 = x - y$

(iv) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

Rule (iii) is very useful when you want to rationalise the denominator of an expression.

Example: $\frac{(2\sqrt{x}-3\sqrt{y})}{(\sqrt{x}-\sqrt{y})} = \frac{(2\sqrt{x}-3\sqrt{y})}{(\sqrt{x}-\sqrt{y})} \times \frac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \frac{2\sqrt{x}\sqrt{x}-3\sqrt{x}\sqrt{y}+2\sqrt{x}\sqrt{y}-3\sqrt{y}\sqrt{y}}{\sqrt{x}\sqrt{x}-\sqrt{y}\sqrt{y}}$

$$= \frac{2x-3y-\sqrt{x}\sqrt{y}}{x-y} = \frac{2x-3y-\sqrt{xy}}{x-y}$$