Rules of probability

Suppose a random experiment was performed n times and the event A occurred $n(A)$ times. The proportion of times event A occurred is $\frac{n(A)}{n}$, i.e., the relative frequency of event A. This might vary a little as n increases but it would tend to some limit as n became very large. This limit is called the probability of the event A and denoted as $P(A)$. This definition is difficult to work with in practice because of the need for a large number of experiments. We can however take a short cut for situations where we have equally likely events. Imagine a fair die being tossed. The sample space is $S = \{1,2,3,4,5,6\}$, i.e., it has 6 outcomes. There is no reason to believe that the probability of any one of the outcomes is different to any other and so we say the probability of each is $\frac{1}{6}$, as there are 6 possible outcomes. If the event of interest had k outcomes out of a sample space of n outcomes then we would assign a probability of $\frac{k}{n}$. **This procedure is called the equally likely principle**.

This analogy prompts the following three fundamental rules of probability.

(i) $0 \le P(A) \le 1$
(ii) $P(S) = 1$
(iii) $P(A \cup B \cup \ldots.) = P(A) + P(B) + \cdots$ for exclusive events $A, B, \ldots.$

These three basic rules can be used to prove the following rules.

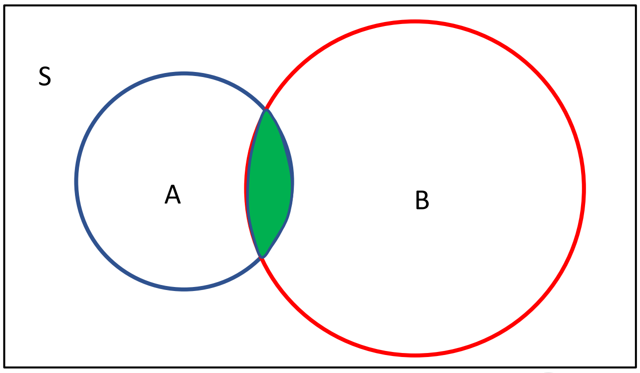
(iv) $P(A') = 1 - P(A)$ [$S = A \cup A'$]
(v) $P(\Phi) = 0$ [$\Phi = S'$]
(vi) $P(A \cap B') = P(A) - P(A \cap B)$ [$A = (A \cap B) \cup (A \cap B')$ Union of mutually exclusive events]
(vii) If $A \subseteq B$ $P(A) \le P(B)$ [$B = A \cup (A' \cap B)$ Union of mutually exclusive events.]
(viii) For any two events A and B $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [$A \cup B = A \cup (A' \cap B)$ Union of two mutually exclusive events.]

Hints of how to prove these results are shown inside the square brackets in red.

Conditional Probability

Probabilities are affected by the information given. For example, if a fair 6 sided die is tossed, the probability of a 6 is $\frac{1}{6}$ since only one outcome out of six corresponds to the event of interest, and together with the equally likely principle gives the result. Now, suppose you are told that the outcome was an even number. This means the outcome was a 2, 4 or 6 and so the probability it was a 6, given this information, is $\frac{1}{3}$.

Suppose we are interested in an event A, but are told that event B has occurred. Succinctly we are interested in the probability that A occurs "given" that event B has occurred. Symbolically this is written as $P(A|B)$, the vertical line reads "given". Consider the diagram alongside. B is the event illustrated by the larger red circle and for the event B to have occurred the outcome must be within B. Those outcomes in B that are also in A are found in the green shaded area and have probability $P(A \cap B)$. Being told that B has occurred essentially makes B our new sample space and so this motivates the definition



$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0 \quad \text{or equivalently} \quad P(A \cap B) = P(A|B)P(B)$$

In our earlier example the event A is that a 6 occurs when a fair die is tossed, and B the event that the outcome is even. We can write $P(A) = \frac{1}{6}$, $P(B) = \frac{3}{6}$ and $P(A \cap B) = \frac{1}{6}$ since the event $A \cap B$ is (having a 6) and (having an even score) = (having a 6). The probability is therefore $\frac{1}{6}$. Thus

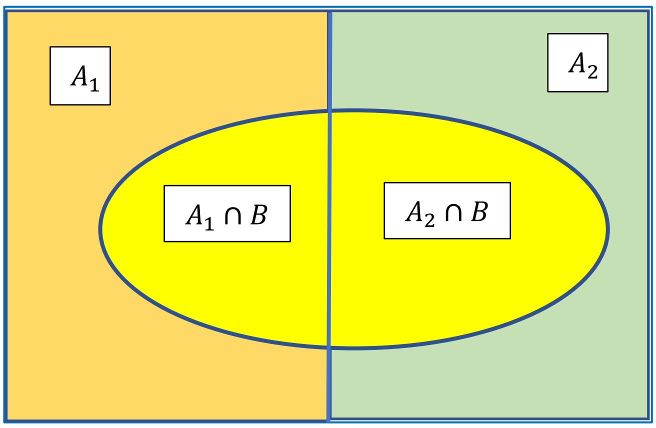
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{1}{6}\right)}{\left(\frac{3}{6}\right)} = \frac{1}{3}$$ the result we deduced earlier.

Law of Total Probability

Suppose the sample space can be divided into two mutually exclusive and exhaustive events A_1 and A_2 and B is any other event. This situation is illustrated in the figure alongside, where the event B is shown in yellow.

Because A_1 and A_2 are exhaustive $A_1 \cup A_2 = S$. They are also exclusive and so $A_1 \cap A_2 = \Phi$. These are the two outside boxes. The event B is all outcomes within the ellipse.

Since A_1 and A_2 are exclusive events so are the events $A_1 \cap B$ and $A_2 \cap B$. It follows that we can write



$$P(B) = P\big((A_1 \cap B) \cup (A_2 \cap B)\big)$$

$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$. This is a useful result in itself, but we can also write

$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$ using conditional probabilities.

Generally, if $A_1, \dots \dots, A_n$ are a set of mutually exclusive and exhaustive events we can write

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

This last result is the **Law of Total Probability**.

Bayes' Theorem

Let $A_1, \ldots \ldots, A_n$ be a set of mutually exclusive and exhaustive events and B another event of interest. If the event B is observed then the natural question is "what is the probability that it was event A_i that was responsible?", i.e., we need to find $P(A_i|B)$. Bayes' result is

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Example: Suppose there are 3 factories producing the same item and that the items were pooled together as one batch before they are sold. Factory 1, produced 50% of the output and factories 2 and 3, 25% each. An item could be classified as a defective and from past records the probabilities of a defective for factories 1,2 and 3 are 0.02, 0.03 and 0.04 respectively. Given that an item was found to be defective what is the probability that it came from factory 2? The information given enables us to write

$$P(D|A_1) = 0.02,\ P(D|A_2) = 0.03,\ P(D|A_3) = 0.04 \quad P(A_1) = 0.5, P(A_2) = P(3) = 0.25$$

where D denotes the event that an item selected at random is defective. The events $A_1, A_2,$ and A_3 are mutually exclusive and exhaustive since each item must come from one, and only one factory. Using the Law of Total Probability we can write

$$P(D) = \sum_{i=1}^{3} P(D|A_i)P(A_i) = 0.02 \times 0.5 + 0.03 \times 0.25 + 0.04 \times 0.25 = 0.0275$$

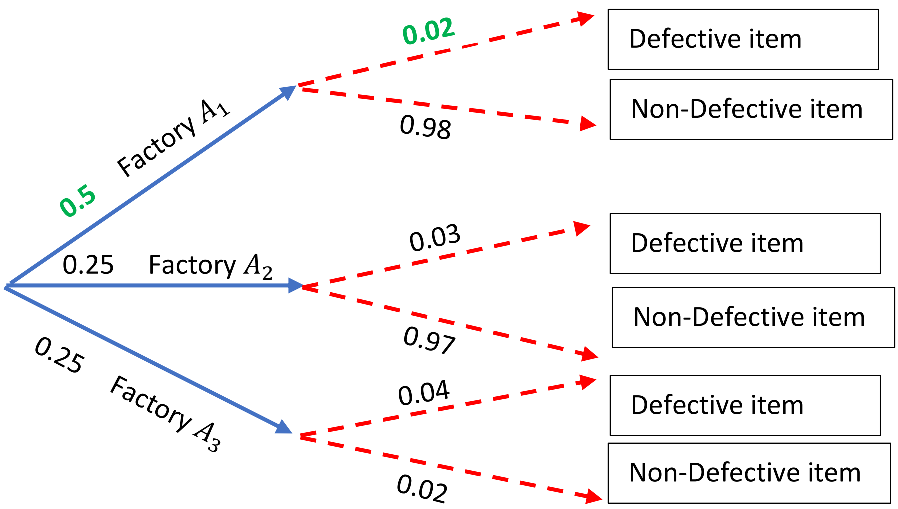
From the definition of conditional probability, we can write

$$P(A_2|D) = \frac{P(A_2 \cap D)}{P(D)} = \frac{P(D|A_2)P(A_2)}{P(D)} = \frac{0.03 \times 0.25}{0.0275} = 0.273$$

i.e., 27.3% of all defective items would come from factory 2.

Tree Diagrams

The above problem can be illustrated by means of a "**Tree Diagram**" where the branches of the tree represent decision points. We start with all of the items together. Firstly, we can assign each to a factory, and then from the given factory we can assign them to be defective or not. The tree diagram is shown alongside. To find the probability of an outcome the probabilities along the branches leading to that outcome should be multiplied. For example, the probability that Factory 1 is chosen and the item is defective is $0.5 \times 0.02 = 0.01$ (product of green numbers). This is underpinned by the conditional probability statement



$$\boldsymbol{P(\text{Defective from Factory } A_1) = P(\text{Defective}|\text{Factory } A_1)P(\text{Factory } A_1) = 0.5 \times 0.02 = 0.01}$$

Independent Events

The events A and B are said to be independent of one another if the outcome of one is unaffected by the outcome of the other. For example, when throwing a fair die twice the outcomes in the second toss are uninfluenced by the outcomes in the first throw.

If this is the case the conditional probability $P(A|B)$ should be the same as $P(A)$ since knowledge of whether B occurs or not does not affect the outcome of A. Thus, to show independence you need to establish the truth of the statement $P(A|B) = P(A)$. From the definition of conditional probability this is equivalent to

$$\frac{P(A \cap B)}{P(B)} = P(A) \quad i.e., \quad P(A \cap B) = P(A)P(B)$$

So, independence is established if $P(A|B) = P(A)$, or equivalently, $P(A \cap B) = P(A)P(B)$.