Distance, Displacement, Speed, Velocity and Acceleration

Suppose, relative to some coordinate system, that a particle is at position $\boldsymbol{r}(t_1)$ at time t_1 and at $\boldsymbol{r}(t_2)$ at time t_2. Then the **average velocity** over the time interval $[t_1, t_2]$ is the rate of change of distance with respect to time which is $\boldsymbol{Average\ Velocity} = \frac{(\boldsymbol{r}(t_2)-\boldsymbol{r}(t_1))}{(t_2-t_1)}$. To obtain the velocity at a point we just let $t_2 \to t_1$. The result is the derivative of $\boldsymbol{r}(t)$ with respect to time t. Thus the velocity at a point is $\boldsymbol{v} = \frac{dr}{dt} = \left(\frac{dx}{dt}\right)\boldsymbol{i} + \left(\frac{dy}{dt}\right)\boldsymbol{j} + \left(\frac{dz}{dt}\right)\boldsymbol{k}.$ i.e., when differentiating a vector, you differentiate it component wise. Similarly, the acceleration is the rate of change of velocity with time, i.e.,

$$\boldsymbol{a} = \frac{dv}{dt} = \left(\frac{dv_x}{dt}\right)\boldsymbol{i} + \left(\frac{dv_y}{dt}\right)\boldsymbol{j} + \left(\frac{dv_z}{dt}\right)\boldsymbol{k}.$$

Bringing these results together we have

Position vector $\boldsymbol{r} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$ m

Velocity vector $\boldsymbol{v} = \frac{dr}{dt} = (v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}) = \left(\left(\frac{dx}{dt}\right)\boldsymbol{i} + \left(\frac{dy}{dt}\right)\boldsymbol{j} + \left(\frac{dz}{dt}\right)\boldsymbol{k}\right)$ m/s

Acceleration vector $\boldsymbol{a} = \frac{dv}{dt} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) = \left(\left(\frac{dv_x}{dt}\right)\boldsymbol{i} + \left(\frac{dv_y}{dt}\right)\boldsymbol{j} + \left(\frac{dv_z}{dt}\right)\boldsymbol{k}\right)$ m/s^2

NB. Notice the basic units for position, velocity and acceleration in SI units. Also note that the dot notation can be used to indicate differentiation with respect to time,

i.e., $\boldsymbol{v} = \dot{\boldsymbol{r}}, \boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}.$

Example:- If the position vector of a particle at time t is $\boldsymbol{r}(t) = (t^2\boldsymbol{i} + 4t^3\boldsymbol{j} + (1 - sint)\boldsymbol{k})$, determine its velocity and acceleration at time t.

Here $\boldsymbol{v} = \frac{dr}{dt} = \left(\frac{d}{dt}(t^2)\right)\boldsymbol{i} + \left(\frac{d}{dt}(4t^3)\right)\boldsymbol{j} + \left(\frac{d}{dt}(1 - sint)\right)\boldsymbol{k}$

$= 2t\boldsymbol{i} + 12t^2\boldsymbol{j} - cost\boldsymbol{k}$

Also $\boldsymbol{a} = \left(\frac{d}{dt}(2t)\right)\boldsymbol{i} + \left(\frac{d}{dt}(12t^2)\right)\boldsymbol{j} - \left(\frac{d}{dt}cost\right)\boldsymbol{k}$

$= 2\boldsymbol{i} + 24t\boldsymbol{j} + sint\boldsymbol{k}$

Example:- If the acceleration of a particle at time t is $\boldsymbol{a} = 2\boldsymbol{i} + 6\boldsymbol{j} + 9\boldsymbol{k}$, determine expressions for the velocity and position at time t.

Since $\boldsymbol{a} = \frac{dv}{dt}$ then $\boldsymbol{v} = \int \boldsymbol{a}dt + \boldsymbol{C},$ where $\boldsymbol{C} = (C_x, C_y, C_z)$ is a vector constant of integration. Hence

$$\boldsymbol{v} = (\int 2dt)\boldsymbol{i} + (\int 6dt)\boldsymbol{j} + (\int 9dt)\boldsymbol{k} + \boldsymbol{C} = 2t\boldsymbol{i} + 6t\boldsymbol{j} + 9t\boldsymbol{k} + \boldsymbol{C}$$

Integrating again gives

$$\boldsymbol{r} = \left(\int 2t dt\right)\boldsymbol{i} + \left(\int 6t dt\right)\boldsymbol{j} + \left(\int 9t dt\right)\boldsymbol{k} + \int \boldsymbol{C}\, dt$$
$$= t^2\boldsymbol{i} + 3t^2\boldsymbol{j} + \frac{9}{2}t^2\boldsymbol{k} + \boldsymbol{C}t + \boldsymbol{B}$$

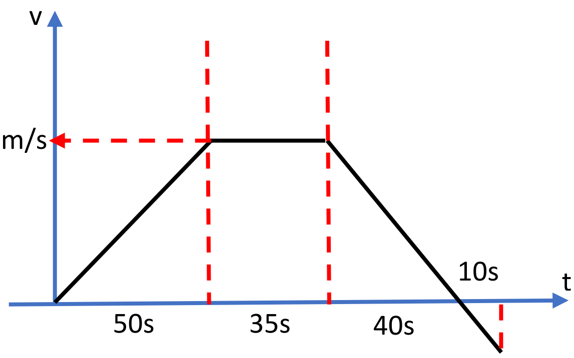
where $\boldsymbol{B} = (B_x, B_y, B_z)$, is a vector constant of integration. As usual **C** and **B** can be determined when boundary conditions are given.

Note **The distance of a particle from the origin** is denoted as $|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$

The **speed of a particle** is $v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, both are **positive scalar quantities**.

Velocity – time plot for 1 dimensional motion.

Suppose we were given the velocity-time plot shown below. Describe its features and determine the total distance travelled. We note the following:-



1. For the first 50 s the particle is travelling with constant acceleration since the plot is linear over this range. The acceleration is the rate of change of velocity and so the acceleration is

$$a = \frac{Change\ in\ vel}{change\ in\ time}$$
$$= \frac{20-0}{50} = 0.4ms^{-2}$$

2. The velocity is constant for the next 35s and so the acceleration is 0 here.
3. The velocity declines to zero over the next 40s and so there is a retardation of $a = \frac{0-20}{40} = -0.5ms^{-2}$. This is also true for the last 10s.
4. Lastly, we note that the area under the velocity time curve is

$$\int_{t_0}^{t_1} v dt = \int_{t_0}^{t_1} \frac{dr}{dt} dt = \int_{t_0}^{t_1} dr = r(t_1) - r(t_0) = \text{Displacement}$$

Distance travelled in the first 125s, whilst the velocity is positive, is

$$\frac{50\times 20}{2} + 35 \times 20 + \frac{20\times 40}{2} = 1600m$$

In the last 10s the velocity is negative, and equal to $-0.5 \times 10 = -5\ ms^{-1}$ after $135s$, and so it is travelling backwards to the origin from 125s. Distance travelled here is $\frac{10\times 5}{2} = 25m.$

In summary

Displacement from the origin $= 1600 - 25 = 1575m$

Distance travelled $= 1600 + 25 = 1625m$

NB a curved section of the curve implies there is a time dependent acceleration.

Displacement-time plot associated with the above velocity-time plot

1. In the first 50s the velocity is proportional to time and is zero when t = 0. Thus $r = \int \frac{20t}{50} dt = 0.2t^2$. This is a quadratic, with a minimum at t=0 and rising to 500m when t = 50s.
2. The velocity is constant(20m/s) for $50 < t < 85$. Therefore $r = 20(t - 50) + 500$ over this range. Thus, when t = 85, $r = 20 \times 35 + 500 = 1200m$.
3. For $85 < t < 135$, $v = -0.5(t - 85) + 20 = -0.5t + 62.5$ and so by integrating

   $r = -\frac{t^2}{4} + 62.5t + C$. Now $r = 1200 \text{ when } t = 85$, hence C = -2306.25.

   This distance is maximised when v = 0, i.e., when t = 125s, the maximum value being

   1600m.

The shape of this displacement time plot is shown below.